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Braided Coadditive Differential Complexes on Quantized Braided Groups

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Braided coadditions of differential complexes on some further generalized quantized braided matrix algebras are constructed. With these coadditions the generalized algebraic systems form a kind of braided (additive) differential Hopf algebras. This is a generalization and unification of some existing results. The coadditions of differential complexes on the usual braided matrices and quantum matrices, etc., can be obtained as special cases.

1. INTRODUCTION

Besides the multiplicative coproducts, in the last few years the additive coproducts (coadditions) on the quantum matrices (groups) and braided matrices (groups), etc., have attracted increasing attention owing to their importance in mathematics and mathematical physics (Majid, 1994a; Meyer, 1995). After the introduction and some studies of differential calculuses on the quantum matrices and braided matrices (Azcarraga *et al.*, 1994; Ogievetsky *et al.*, 1992; Sudbery, 1992, 1993), the coadditions on these differential complexes were also considered by some authors (Isaev and Vladimirov, 1995; Vladimirov, 1994). On the other hand, a kind of more general algebraic systems—the quantized braided matrices (groups) (QBM(G)s)—were proposed by Hlavaty (1994); these contain the usual quantum (Faddeev *et al.*, 1989; Manin, 1988) and braided (Kulish and Sasaki, 1993; Kulish and Sklyanin, 1992; Majid, 1991, 1993) matrices (groups), etc., as special cases (cf. Hlavaty, 1997). Thus the theory of QBM(G) has, among others, a remarkable advantage, enabling us to study the two quite different kinds of noncommutativities (associated, respectively, with quantization and general braid statistics) in a unified way.

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Recently, we have constructed differential calculuses on the QBM(G)s and discussed some of their properties (Gao and Gui, in press). In the present paper, we shall construct braided coadditions of the differential complexes on QBMs in a slightly further generalized form and show that they form a kind of braided (additive) differential Hopf algebra. The so-obtained braided Hopf algebra is a generalization and unification of some existing results, the coadditions of the differential complexes on braided matrices (Isaev and Vladimirov, 1995) and on quantum matrices (Vladimirov, 1994), etc., can be obtained as special cases.

In the discussions of this paper, we need the braidings $\Psi_{A,B} : A \otimes B \to B \otimes A, \ldots$ between the related algebras $A, B, C, \ldots,$ which obey certain rules (Joyal and Street, 1986; Majid, 1994b, and references therein), for example,

$$\Psi_{A,B\otimes C} = \Psi_{A,C}\Psi_{A,B}, \qquad \Psi_{A,\otimes B,C} = \Psi_{A,C}\Psi_{B,C} \tag{1.1}$$

and the braid relation

$$\Psi_{B,C}\Psi_{A,C}\Psi_{A,B} = \Psi_{A,B}\Psi_{A,C}\Psi_{B,C}. \tag{1.2}$$

In general, $\Psi^2 \neq id$. The structure of the braided tensor product algebra $A \otimes B$ involves the braidings in such a way that

$$(a \otimes b)(a' \otimes b') = a\Psi_{B,A}(b \otimes a')b', \qquad a, a' \in A, \ \ b, b' \in B. \tag{1.3}$$

If A is a braided Hopf algebra, then its (braided) antipode S obeys

$$S(aa') = \cdot\Psi(Sa \otimes Sa'). \tag{1.4}$$

In Section 2, we first recall some results of QBM(G)s (Gao and Gui, 1997; Hlavaty, 1994) and related differential complexes obtained recently (Gao and Gui, in press), then we give some simple generalization of them. In Section 3, some braided coadditions of differential complexes on the generalized QBMs are constructed and the braided Hopf algebra structures of the obtained differential bialgebras are demonstrated. Some reduced cases are given in Section 4. It is shown that the braided (additive) differential Hopf algebras on the braided groups (Isaev and Vladimirov, 1995) and quantum groups (Vladimirov, 1994), etc., can be obtained as special cases of the scheme in Section 3.

2. DIFFERENTIAL CALCULUSES ON QBM(G)S AND THEIR GENERALIZATION

The QBM algebra, denoted by *A(R, Z)*, is defined as follows (Hlavaty, 1994). Let $T = \{T^i_j\}^N_{i,j=1}$ be a matrix of N^2 elements T^i_j and $R, Z \in M_N(\mathbf{C}) \otimes M_N(\mathbf{C})$ be an R-matrix pair satisfying the set of quantum Yang–Baxter-type equations

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{23}R_{13}R_{12}, \qquad Z_{12}Z_{13}Z_{23} = Z_{23}Z_{13}Z_{12}, \\ R_{12}Z_{13}Z_{23} &= Z_{23}Z_{13}R_{12}, \qquad Z_{12}Z_{13}R_{23} = R_{23}Z_{13}Z_{12}, \end{aligned} \tag{2.1}$$

then $A(R, Z)$ is generated by $\{T^i_j, 1\}$ with the following algebra relation

$$R_{12}Z_{12}^{-1}T_1Z_{12}T_2 = Z_{21}^{-1}T_2Z_{21}T_1R_{12}. \tag{2.2}$$

Here the R and Z are both assumed to be invertible.

A matrix solution R of the quantum Yang–Baxter equation (QYBE) is termed Hecke type if it satisfies

$$(PR - q)(PR + q^{-1}) = 0, \qquad \text{or} \qquad PRPR = 1 + \lambda PR \tag{2.3}$$

for suitable q and $\lambda = q - q^{-1}$. Where P is the usual permutation matrix.

Proposition 2.1 (Gao and Gui, 1997). *Defining* $R^{(n)} \equiv (ZP)^n R(Z^{-1}P)^n$, *if* (R, Z) *is an R-matrix pair satisfying (2.1), then* $(R^{(n)}, Z)$ *too satisfies (2.1) for each integer* $n = 0, \pm 1, \pm 2, \ldots$. *Moreover, if R is Hecke type, then so is* $R^{(n)}$.

Recently, we had constructed the differential calculuses on $A(R, Z)$, the algebra relations of which are (Gao and Gui, in press)

$$\begin{aligned} R_{12}Z_{12}^{-1}T_1Z_{12}T_2 &= Z_{21}^{-1}T_2Z_{21}T_1R_{12}, \\ R_{21}^{-1}Z_{12}^{-1}T_1Z_{12}dT_2 &= Z_{21}^{-1}dT_2Z_{21}T_1R_{12}, \\ R_{21}^{-1}Z_{12}^{-1}dT_1Z_{12}dT_2 &= -Z_{21}^{-1}dT_2Z_{21}dT_1R_{12}, \end{aligned} \tag{2.4}$$

where d is the external differentiation obeying $d^2 = 0$ and the usual (graded) Leibnitz rule. The differential algebra given by (2.4) will be denoted by $\Omega_{A(R,Z)}$.

In the following part of this section, we give some simple generalization of $\Omega_{A(R,Z)}$.

Definition 2.2. Consider an ordered triple of numerical matrices $Q, R, Z \in M_N \otimes M_N$, if Q, R are both invertible and Q, R, Z satisfy the following mixed QYBEs

$$\begin{aligned} Q_{12}Q_{13}Q_{23} &= Q_{23}Q_{13}Q_{12}, \qquad & R_{12}R_{13}R_{23} &= R_{23}R_{13}R_{12}, \\ Z_{12}Z_{13}Q_{23} &= Q_{23}Z_{13}Z_{12}, \qquad & R_{12}Z_{13}Z_{23} &= Z_{23}Z_{13}R_{12}, \end{aligned} \tag{2.5}$$

then we call this ordered triple an *R-matrix triple* and denote it by (Q, R, Z).

Proposition 2.3. *Let* (Q, R, Z) *be an R-matrix triple, let also Q, R both be Hecke type and Z have the second inverse* $\tilde{Z} \equiv ((Z^{t_2})^{-1})^{t_2}$ (t_2 *denotes transposition in the second factor*). *Then a differential algebra* $\Omega_{A(Q,R,Z)}$ *can be constructed as follows: it is generated by* $T = \{T^i_j\}_{i,j=1}^N$, $dT = \{dT^i_j\}_{i,j=1}^N$, 1 *and has the algebra relations*

$$Q_{21}T_1Z_{12}T_2 = T_2Z_{21}T_1R_{12}, \tag{2.6a}$$

$$Q_{12}^{-1}T_1Z_{12}dT_2 = dT_2Z_{21}T_1R_{12}, \tag{2.6b}$$

$$Q_{12}^{-1}dT_1Z_{12}dT_2 = -dT_2Z_{21}dT_1R_{12}. \tag{2.6c}$$

Proof: The subalgebra (2.6a) has been given by Friedel and Maillet (1991) in a somewhat more general form. However, here the $\Omega_{A(Q,R,Z)}$ contains also the differential forms $\{dT^i_j\}$ as a part of generators, so we have to check that the relations (2.6a)–(2.6c) as a whole can consistently define an associative algebra. To this end, we consider, for example, the expression containing triple of the generators as

$$T_1 Z_{12} dT_2 Z_{13} Z_{23} dT_3. \tag{2.7}$$

We involve Z in (2.7) so that the algebra relations (2.6) can be conveniently used, and this does not lose the generality because of the existence of $\tilde{Z}$. Transposing (2.7) in two ways, from (2.6) we obtain

$$\begin{aligned}
\underline{T_1 Z_{12} dT_2} Z_{13} Z_{23} dT_3 &= Q_{12} dT_2 Z_{21} T_1 \underline{R_{12} Z_{13} Z_{23}} dT_3\\
&= Q_{12} dT_2 Z_{21} Z_{23} \underline{T_1 Z_{13} dT_3} R_{12}\\
&= Q_{12} dT_2 \underline{Z_{21} Z_{23} Q_{13}} dT_3 Z_{31} T_1 R_{13} R_{12}\\
&= Q_{12} Q_{13} \underline{dT_2 Z_{23} dT_3} Z_{21} Z_{31} T_1 R_{13} R_{12}\\
&= -Q_{12} Q_{13} Q_{23} dT_3 Z_{32} dT_2 \underline{R_{23} Z_{21} Z_{31}} T_1 R_{13} R_{12}\\
&= -Q_{12} Q_{13} Q_{23} dT_3 Z_{32} dT_2 Z_{31} Z_{21} T_1 R_{23} R_{13} R_{12},\\
T_1 Z_{12} \underline{dT_2 Z_{13} Z_{23} dT_3} &= -T_1 \underline{Z_{12} Z_{13} Q_{23}} dT_3 Z_{32} dT_2 R_{23}\\
&= -Q_{23} \underline{T_1 Z_{13} dT_3} Z_{12} Z_{32} dT_2 R_{23}\\
&= -Q_{23} Q_{13} dT_3 Z_{31} T_1 \underline{R_{13} Z_{12} Z_{32}} dT_2 R_{23}\\
&= -Q_{23} Q_{13} dT_3 Z_{31} Z_{32} \underline{T_1 Z_{12} dT_2} R_{13} R_{23}\\
&= -Q_{23} Q_{13} dT_3 \underline{Z_{31} Z_{32} Q_{12}} dT_2 Z_{21} T_1 R_{12} R_{13} R_{23}\\
&= -Q_{23} Q_{13} Q_{12} dT_3 Z_{32} dT_2 Z_{31} Z_{21} T_1 R_{12} R_{13} R_{23}.
\end{aligned}$$

The two results are equal from the QYBEs of Q and R. In the above calculations we have used (2.5), (2.6), and their index-relabelled forms many times to the underlined parts in each expression to obtain the next expression. For other triple products of generators, the calculations are similar. So these do not impose any additional relation on T, dT and thus the differential algebra $\Omega_{A(Q,R,Z)}$ is consistently defined. □

Taking $Q_{12} = Z_{12} R_{21} Z_{21}^{-1}$, then from Proposition 2.1 the $\Omega_{A(Q,R,Z)}$ is reduced to the $\Omega_{A(R,Z)}$ given by (2.4). So $\Omega_{A(R,Z)}$ is a special case of $\Omega_{A(Q,R,Z)}$.

3. BRAIDED ADDITIVE COPRODUCT (COADDITION) ON $\Omega_{A(Q,R,Z)}$

In this section we show that the differential algebra $\Omega_{A(Q,R,Z)}$ given by (2.6) admits some braided coaddition and with this coaddition $\Omega_{A(Q,R,Z)}$ becomes a braided Hopf algebra.

Theorem 3.1. *Let* (Q, R, Z) *be an R-matrix triple, let also* Q, R *be Hecke type and* Z *have the second inverse* $\tilde{Z}$; *then the differential algebra* $\Omega_{A(Q,R,Z)}$ *given by (2.6) forms a braided Hopf algebra if it is provided with the additive coproduct*

$$\begin{aligned} \Delta(T) &= T\otimes 1 + 1\otimes T \equiv T + \tilde{T}, \\ \Delta(dT) &= dT\otimes 1 + 1\otimes dT \equiv dT + d\tilde{T}, \end{aligned} \tag{3.1}$$

the counit, antipode

$$\begin{aligned} &\varepsilon(1) = 1, \qquad \varepsilon(T) = \varepsilon(dT) = 0, \\ &S(1) = 1, \qquad S(T) = -T, \qquad S(dT) = -dT, \end{aligned} \tag{3.2}$$

and any one of the following four braiding relations

$$\begin{aligned} \tilde{T}_1 Z_{12} T_2 &= Q_{12} T_2 Z_{21} \tilde{T}_1 R_{12}, \\ \tilde{T}_1 Z_{12} dT_2 &= Q_{12} dT_2 Z_{21} \tilde{T}_1 R_{12} + \lambda P_{12} T_2 Z_{21} d\tilde{T}_1 R_{12}, \\ d\tilde{T}_1 Z_{12} T_2 &= Q_{21}^{-1} T_2 Z_{21} d\tilde{T}_1 R_{12}, \\ d\tilde{T}_1 Z_{12} dT_2 &= -Q_{21}^{-1} dT_2 Z_{21} d\tilde{T}_1 R_{12}; \end{aligned} \tag{3.3a}$$

$$\begin{aligned} \tilde{T}_1 Z_{12} T_2 &= Q_{21}^{-1} T_2 Z_{21} \tilde{T}_1 R_{12}^{-1}, \\ \tilde{T}_1 Z_{12} dT_2 &= Q_{12} dT_2 Z_{21} \tilde{T}_1 R_{12}^{-1}, \\ d\tilde{T}_1 Z_{12} T_2 &= Q_{21}^{-1} T_2 Z_{21} d\tilde{T}_1 R_{12}^{-1} - \lambda P_{12} dT_2 Z_{21} \tilde{T}_1 R_{12}^{-1}, \\ d\tilde{T}_1 Z_{12} dT_2 &= -Q_{12} dT_2 Z_{21} d\tilde{T}_1 R_{12}^{-1}; \end{aligned} \tag{3.3b}$$

$$\begin{aligned} \tilde{T}_1 Z_{12} T_2 &= Q_{12} T_2 Z_{21} \tilde{T}_1 R_{12}, \\ \tilde{T}_1 Z_{12} dT_2 &= Q_{12} dT_2 Z_{21} \tilde{T}_1 R_{12} + \lambda Q_{12} T_2 Z_{21} d\tilde{T}_1 P_{12}, \\ d\tilde{T}_1 Z_{12} T_2 &= Q_{12} T_2 Z_{21} d\tilde{T}_1 R_{21}^{-1}, \\ d\tilde{T}_1 Z_{12} dT_2 &= -Q_{12} dT_2 Z_{21} d\tilde{T}_1 R_{21}^{-1}; \end{aligned} \tag{3.3c}$$

$$\begin{aligned} \tilde{T}_1 Z_{12} T_2 &= Q_{21}^{-1} T_2 Z_{21} \tilde{T}_1 R_{21}^{-1}, \\ \tilde{T}_1 Z_{12} dT_2 &= Q_{21}^{-1} dT_2 Z_{21} \tilde{T}_1 R_{12}, \\ d\tilde{T}_1 Z_{12} T_2 &= Q_{21}^{-1} T_2 Z_{21} d\tilde{T}_1 R_{12}^{-1} - \lambda Q_{21}^{-1} dT_2 Z_{21} \tilde{T}_1 P_{12}, \\ d\tilde{T}_1 Z_{12} dT_2 &= -Q_{21}^{-1} dT_2 Z_{21} d\tilde{T}_1 R_{12}. \end{aligned} \tag{3.3d}$$

Proof: The discussion of Vladimirov (1994) is one of our motivations. In this paper, we shall write braidings in two ways: use Ψ as in Section 1 or use symbols with tilde (cf. (3.1)) and omit writing Ψ and the tensor product $\otimes$ as in (3.3). Here we introduce the following notation:

$$\varphi = \begin{pmatrix} T \\ dT \end{pmatrix}, \qquad \eta = (T \quad dT), \tag{3.4}$$

$$V_{12} = \begin{pmatrix} Q_{21}^{-1} & 0 & 0 & 0 \\ 0 & Q_{21}^{-1} & -\lambda P_{12} & 0 \\ 0 & 0 & Q_{12} & 0 \\ 0 & 0 & 0 & -Q_{12} \end{pmatrix}, \qquad \bar{V}_{12} = \begin{pmatrix} Q_{12} & 0 & 0 & 0 \\ 0 & Q_{12} & \lambda P_{12} & 0 \\ 0 & 0 & Q_{21}^{-1} & 0 \\ 0 & 0 & 0 & -Q_{21}^{-1} \end{pmatrix}, \tag{3.5a}$$

$$W_{12} = \begin{pmatrix} R_{21}^{-1} & 0 & 0 & 0 \\ 0 & R_{21}^{-1} & 0 & 0 \\ 0 & -\lambda P_{12} & R_{12} & 0 \\ 0 & 0 & 0 & -R_{12} \end{pmatrix}, \qquad \bar{W}_{12} = \begin{pmatrix} R_{12} & 0 & 0 & 0 \\ 0 & R_{12} & 0 & 0 \\ 0 & \lambda P_{12} & R_{21}^{-1} & 0 \\ 0 & 0 & 0 & -R_{21}^{-1} \end{pmatrix}. \tag{3.5b}$$

Then considering the Hecke property of Q, R, the algebra relations (2.6) can be rewritten in several forms as follows

$$\varphi_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}} = V_{12}^{\mathrm{II\,I}} \varphi_2^{\mathrm{II}} Z_{21} \varphi_1^{\mathrm{I}} R_{12} \tag{3.6a}$$

$$= \bar{V}_{12}^{\mathrm{I\,II}} \varphi_2^{\mathrm{II}} Z_{21} \varphi_1^{\mathrm{I}} R_{21}^{-1}, \tag{3.6b}$$

$$\eta_1^{\mathrm{I}} Z_{12} \eta_2^{\mathrm{II}} = Q_{12} \eta_2^{\mathrm{II}} Z_{21} \eta_1^{\mathrm{I}} W_{12}^{\mathrm{II\,I}} \tag{3.6c}$$

$$= Q_{21}^{-1} \eta_2^{\mathrm{II}} Z_{21} \eta_1^{\mathrm{I}} \bar{W}_{12}^{\mathrm{I\,II}}. \tag{3.6d}$$

Where the indices I, II, etc., enumerate the "super" matrix and vector spaces whose elements themselves are matrices (such as the above φ, V, etc.), we use the Roman numerals and write them as superscripts to distinguish them from the usual Arabic numeral subscripts 1, 2, etc.

By the aid of (3.6a)–(3.6d), we now show that the differential algebra $\Omega_{A(Q,R,Z)}$ admits the additive coproduct (3.1) with any one of the following braidings

$$\tilde{\varphi}_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}} = \bar{V}_{12}^{\mathrm{I\,II}} \varphi_2^{\mathrm{II}} Z_{21} \tilde{\varphi}_1^{\mathrm{I}} R_{12}, \tag{3.7a}$$

$$\tilde{\varphi}_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}} = V_{12}^{\mathrm{II\,I}} \varphi_2^{\mathrm{II}} Z_{21} \tilde{\varphi}_1^{\mathrm{I}} R_{12}^{-1}, \tag{3.7b}$$

$$\tilde{\eta}_1^{\mathrm{I}} Z_{12} \eta_2^{\mathrm{II}} = Q_{12} \eta_2^{\mathrm{II}} Z_{21} \tilde{\eta}_1^{\mathrm{I}} \bar{W}_{12}^{\mathrm{I\,II}}, \tag{3.7c}$$

$$\tilde{\eta}_1^{\mathrm{I}} Z_{12} \eta_2^{\mathrm{II}} = Q_{12}^{-1} \eta_2^{\mathrm{II}} Z_{21} \tilde{\eta}_1^{\mathrm{I}} W_{12}^{\mathrm{II\,I}} \tag{3.7d}$$

We remark that the component forms of (3.7a)–(3.7d) are just the braidings (3.3a)–(3.3d) respectively, and the braidings (3.7a) and (3.7b), (3.7c) and (3.7d) (equivalently (3.3a) and (3.3b), (3.3c) and (3.3d)) are inverses of each other, respectively.

First, (3.7a)–(3.7d) obey the axioms of braiding. For example, considering (3.7a) we have

$$
\begin{aligned}
&(\Psi \otimes id)(id \otimes \Psi)(\Psi \otimes id)\big(\varphi_1^{\mathrm{I}} \otimes Z_{12}\varphi_2^{\mathrm{II}} \otimes Z_{13}Z_{23}\varphi_3^{\mathrm{III}}\big)\\
&\quad= (\Psi \otimes id)(id \otimes \Psi)\big(\bar{V}_{12}^{\mathrm{I\,II}}\varphi_2^{\mathrm{II}} \otimes Z_{21}\varphi_1^{\mathrm{I}}R_{12} \otimes Z_{13}Z_{23}\varphi_3^{\mathrm{III}}\big)\\
&\quad= (\Psi \otimes id)(id \otimes \Psi)\big(\bar{V}_{12}^{\mathrm{I\,II}}\varphi_2^{\mathrm{II}} \otimes Z_{21}Z_{23}\varphi_1^{\mathrm{I}} \otimes Z_{13}\varphi_3^{\mathrm{III}}R_{12}\big)\\
&\quad= (\Psi \otimes id)\big(\bar{V}_{12}^{\mathrm{I\,II}}\varphi_2^{\mathrm{II}} \otimes Z_{21}Z_{23}\bar{V}_{13}^{\mathrm{I\,III}}\varphi_3^{\mathrm{III}} \otimes Z_{31}\varphi_1^{\mathrm{I}}R_{13}R_{12}\big)\\
&\quad= (\Psi \otimes id)\big(\bar{V}_{12}^{\mathrm{I\,II}}\bar{V}_{13}^{\mathrm{I\,III}}\varphi_2^{\mathrm{II}} \otimes Z_{23}\varphi_3^{\mathrm{III}} \otimes Z_{21}Z_{31}\varphi_1^{\mathrm{I}}R_{13}R_{12}\big)\\
&\quad= \bar{V}_{12}^{\mathrm{I\,II}}\bar{V}_{13}^{\mathrm{I\,III}}\bar{V}_{23}^{\mathrm{II\,III}}\varphi_3^{\mathrm{III}} \otimes Z_{32}\varphi_2^{\mathrm{II}} \otimes R_{23}Z_{21}Z_{31}\varphi_1^{\mathrm{I}}R_{13}R_{12}\\
&\quad= \bar{V}_{12}^{\mathrm{I\,II}}\bar{V}_{13}^{\mathrm{I\,III}}\bar{V}_{23}^{\mathrm{II\,III}}\varphi_3^{\mathrm{III}} \otimes Z_{32}\varphi_2^{\mathrm{II}} \otimes Z_{31}Z_{21}\varphi_1^{\mathrm{I}}R_{23}R_{13}R_{12},
\end{aligned}
$$

on the other hand

$$
\begin{aligned}
&(id \otimes \Psi)(\Psi \otimes id)(id \otimes \Psi)\big(\varphi_1^{\mathrm{I}} \otimes Z_{12}\varphi_2^{\mathrm{II}} \otimes Z_{13}Z_{23}\varphi_3^{\mathrm{III}}\big)\\
&\quad= (id \otimes \Psi)(\Psi \otimes id)\big(\varphi_1^{\mathrm{I}} \otimes Z_{12}Z_{13}\bar{V}_{23}^{\mathrm{II\,III}}\varphi_3^{\mathrm{III}} \otimes Z_{32}\varphi_2^{\mathrm{II}}R_{23}\big)\\
&\quad= (id \otimes \Psi)(\Psi \otimes id)\big(\bar{V}_{23}^{\mathrm{II\,III}}\varphi_1^{\mathrm{I}} \otimes Z_{13}\varphi_3^{\mathrm{III}} \otimes Z_{12}Z_{32}\varphi_2^{\mathrm{II}}R_{23}\big)\\
&\quad= (id \otimes \Psi)\big(\bar{V}_{23}^{\mathrm{II\,III}}\bar{V}_{13}^{\mathrm{I\,III}}\varphi_3^{\mathrm{III}} \otimes Z_{31}\varphi_1^{\mathrm{I}} \otimes R_{13}Z_{12}Z_{32}\varphi_2^{\mathrm{II}}R_{23}\big)\\
&\quad= (id \otimes \Psi)\big(\bar{V}_{23}^{\mathrm{II\,III}}\bar{V}_{13}^{\mathrm{I\,III}}\varphi_3^{\mathrm{III}} \otimes Z_{31}Z_{32}\varphi_1^{\mathrm{I}} \otimes Z_{12}\varphi_2^{\mathrm{II}}R_{13}R_{23}\big)\\
&\quad= \bar{V}_{23}^{\mathrm{II\,III}}\bar{V}_{13}^{\mathrm{I\,III}}\varphi_3^{\mathrm{III}} \otimes Z_{31}Z_{32}\bar{V}_{12}^{\mathrm{I\,II}}\varphi_2^{\mathrm{II}} \otimes Z_{21}\varphi_1^{\mathrm{I}}R_{12}R_{13}R_{23}\\
&\quad= \bar{V}_{23}^{\mathrm{II\,III}}\bar{V}_{13}^{\mathrm{I\,III}}\bar{V}_{12}^{\mathrm{I\,II}}\varphi_3^{\mathrm{III}} \otimes Z_{32}\varphi_2^{\mathrm{II}} \otimes Z_{31}Z_{21}\varphi_1^{\mathrm{I}}R_{12}R_{13}R_{23}.
\end{aligned}
$$

These two results are equal owing to (2.5). In the above calculations, we have used (2.5), (3.7a), (3.5a), and the equations $P_{12}Z_{13}Z_{23} = Z_{23}Z_{13}P_{12}$, $Z_{12}Z_{13}P_{23} = P_{23}Z_{13}Z_{12}$ many times. For the other braidings (3.7b)–(3.7d) the calculations are similar.

Next, the extension of the braidings to products is considered. For instance, from (3.6a) and (3.7a) we deduce, on the one hand,

$$
\begin{aligned}
\underline{\tilde{\varphi}_1^{\mathrm{I}}Z_{12}Z_{13}\big(\varphi_2^{\mathrm{II}}}Z_{23}\varphi_3^{\mathrm{III}}\big) &= \bar{V}_{12}^{\mathrm{I\,II}}\varphi_2^{\mathrm{II}}Z_{21}\tilde{\varphi}_1^{\mathrm{I}}\underline{R_{12}Z_{13}Z_{23}}\varphi_3^{\mathrm{III}}\\
&= \bar{V}_{12}^{\mathrm{I\,II}}\varphi_2^{\mathrm{II}}Z_{21}Z_{23}\underline{\tilde{\varphi}_1^{\mathrm{I}}Z_{13}\varphi_3^{\mathrm{III}}}R_{12}
\end{aligned}
$$

$$= \bar{V}_{12}^{\mathrm{I\,II}} \varphi_2^{\mathrm{II}} \underline{Z_{21} Z_{23} \bar{V}_{13}^{\mathrm{I\,III}}} \varphi_3^{\mathrm{III}} Z_{31} \tilde{\varphi}_1^{\mathrm{I}} R_{13} R_{12}$$
$$= \bar{V}_{12}^{\mathrm{I\,II}} \bar{V}_{13}^{\mathrm{I\,III}} \left(\varphi_2^{\mathrm{II}} Z_{23} \varphi_3^{\mathrm{III}}\right) Z_{21} Z_{31} \tilde{\varphi}_1^{\mathrm{I}} R_{13} R_{12},$$

and on the other hand,

$$\begin{aligned}
&\tilde{\varphi}_1^{\mathrm{I}} \underline{Z_{12} Z_{13} \left(V_{23}^{\mathrm{III\,II}}\right.} \varphi_3^{\mathrm{III}} Z_{32} \varphi_2^{\mathrm{II}} R_{23}\big) \\
&= V_{23}^{\mathrm{III\,II}} \underline{\tilde{\varphi}_1^{\mathrm{I}} Z_{13} \varphi_3^{\mathrm{III}}} Z_{12} Z_{32} \varphi_2^{\mathrm{II}} R_{23} \\
&= V_{23}^{\mathrm{III\,II}} \bar{V}_{13}^{\mathrm{I\,III}} \varphi_3^{\mathrm{III}} Z_{31} \tilde{\varphi}_1^{\mathrm{I}} \underline{R_{13} Z_{12} Z_{32}} \varphi_2^{\mathrm{II}} R_{23} \\
&= V_{23}^{\mathrm{III\,II}} \bar{V}_{13}^{\mathrm{I\,III}} \varphi_3^{\mathrm{III}} Z_{31} Z_{32} \underline{\tilde{\varphi}_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}}} R_{13} R_{23} \\
&= V_{23}^{\mathrm{III\,II}} \bar{V}_{13}^{\mathrm{I\,III}} \varphi_3^{\mathrm{III}} \underline{Z_{31} Z_{32} \bar{V}_{12}^{\mathrm{I\,II}}} \varphi_2^{\mathrm{II}} Z_{21} \tilde{\varphi}_1^{\mathrm{I}} \underline{R_{12} R_{13} R_{23}} \\
&= \underline{V_{23}^{\mathrm{III\,II}} \bar{V}_{13}^{\mathrm{I\,III}} \bar{V}_{12}^{\mathrm{I\,II}}} \varphi_3^{\mathrm{III}} Z_{32} \varphi_2^{\mathrm{II}} \underline{Z_{31} Z_{21} R_{23}} \tilde{\varphi}_1^{\mathrm{I}} R_{13} R_{12} \\
&= \bar{V}_{12}^{\mathrm{I\,II}} \bar{V}_{13}^{\mathrm{I\,III}} \left(V_{23}^{\mathrm{III\,II}} \varphi_3^{\mathrm{III}} Z_{32} \varphi_2^{\mathrm{II}} R_{23}\right) Z_{21} Z_{31} \tilde{\varphi}_1^{\mathrm{I}} R_{13} R_{12}.
\end{aligned}$$

The two results are equal by (3.6a). In the above calculations, we have used (2.5) and QYBEs about V and $\bar{V}$ obtained from (2.5). The consistency of the extension to the high-order products and for the other braidings (3.7b)–(3.7d) can be verified in entirely similar ways. Hence the braidings (3.7a)–(3.7d) are all well defined and functorial with respect to the product.

Moreover, we extend Δ in (3.1) to products in such a way that it is a homomorphism to the baided tensor product like (1.3). This is consistent because, for example, for (3.6a) we have

$$\begin{aligned}
\Delta\left(\varphi_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}}\right) &= \left(\varphi_1^{\mathrm{I}} + \tilde{\varphi}_1^{\mathrm{I}}\right) Z_{12} \left(\varphi_2^{\mathrm{II}} + \tilde{\varphi}_2^{\mathrm{II}}\right) \\
&= \varphi_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}} + \varphi_1^{\mathrm{I}} Z_{12} \tilde{\varphi}_2^{\mathrm{II}} + \tilde{\varphi}_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}} + \tilde{\varphi}_1^{\mathrm{I}} Z_{12} \tilde{\varphi}_2^{\mathrm{II}} \\
&= \varphi_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}} + \varphi_1^{\mathrm{I}} Z_{12} \tilde{\varphi}_2^{\mathrm{II}} + \bar{V}_{12}^{\mathrm{I\,II}} \varphi_2^{\mathrm{II}} Z_{21} \tilde{\varphi}_1^{\mathrm{I}} R_{12} + \tilde{\varphi}_1^{\mathrm{I}} Z_{12} \tilde{\varphi}_2^{\mathrm{II}},
\end{aligned}$$

$$\begin{aligned}
\Delta\left(V_{12}^{\mathrm{II\,I}} \varphi_2^{\mathrm{II}} Z_{21} \varphi_1^{\mathrm{I}} R_{12}\right) &= V_{12}^{\mathrm{II\,I}} \left(\varphi_2^{\mathrm{II}} + \tilde{\varphi}_2^{\mathrm{II}}\right) Z_{21} \left(\varphi_1^{\mathrm{I}} + \tilde{\varphi}_1^{\mathrm{I}}\right) R_{12} \\
&= V_{12}^{\mathrm{II\,I}} \varphi_2^{\mathrm{II}} Z_{21} \varphi_1^{\mathrm{I}} R_{12} + V_{12}^{\mathrm{II\,I}} \varphi_2^{\mathrm{II}} Z_{21} \tilde{\varphi}_1^{\mathrm{I}} R_{12} \\
&\quad + V_{12}^{\mathrm{II\,I}} \tilde{\varphi}_2^{\mathrm{II}} Z_{21} \varphi_1^{\mathrm{I}} R_{12} + V_{12}^{\mathrm{II\,I}} \tilde{\varphi}_2^{\mathrm{II}} Z_{21} \tilde{\varphi}_1^{\mathrm{I}} R_{12} \\
&= V_{12}^{\mathrm{II\,I}} \varphi_2^{\mathrm{II}} Z_{21} \varphi_1^{\mathrm{I}} R_{12} + V_{12}^{\mathrm{II\,I}} \varphi_2^{\mathrm{II}} Z_{21} \tilde{\varphi}_1^{\mathrm{I}} R_{12} \\
&\quad + V_{12}^{\mathrm{II\,I}} \bar{V}_{21}^{\mathrm{II\,I}} \varphi_1^{\mathrm{I}} Z_{12} \tilde{\varphi}_2^{\mathrm{II}} R_{21} R_{12} + V_{12}^{\mathrm{II\,I}} \tilde{\varphi}_2^{\mathrm{II}} Z_{21} \tilde{\varphi}_1^{\mathrm{I}} R_{12} \\
&= V_{12}^{\mathrm{II\,I}} \varphi_2^{\mathrm{II}} Z_{21} \varphi_1^{\mathrm{I}} R_{12} + V_{12}^{\mathrm{II\,I}} \varphi_2^{\mathrm{II}} Z_{21} \tilde{\varphi}_1^{\mathrm{I}} R_{12} \\
&\quad + \varphi_1^{\mathrm{I}} Z_{12} \tilde{\varphi}_2^{\mathrm{II}} (1 + \lambda P_{12} R_{12}) + V_{12}^{\mathrm{II\,I}} \tilde{\varphi}_2^{\mathrm{II}} Z_{21} \tilde{\varphi}_1^{\mathrm{I}} R_{12},
\end{aligned}$$

the two results are equal owing to (3.6a) and the relation $\bar{V}_{12}^{\mathrm{I\,II}} = V_{12}^{\mathrm{II\,I}} + \lambda P_{12} P^{\mathrm{I\,II}}$ obtained from (3.5a). In the above calculations we have also used $V_{12}^{\mathrm{I\,II}} V_{21}^{\mathrm{I\,II}} = E$ (unit matrix) and the Hecke property of Q, R. So Δ is well defined. It is trivial to see that the braidings (3.7) (or equivalently (3.3)) are functorial with respect to the coproduct Δ.

Finally, for a braided Hopf algebra, the antipode S by definition is braided antimultiplicative in the sense of (1.4). Then we can calculate, for example, for (3.6a) and (3.7a), that

$$S\big(\varphi_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}}\big) = \cdot\Psi\big(S\big(\varphi_1^{\mathrm{I}}\big) \otimes Z_{12} S\big(\varphi_2^{\mathrm{II}}\big)\big) = \bar{V}_{12}^{\mathrm{I\,II}} \varphi_2^{\mathrm{II}} Z_{21} \varphi_1^{\mathrm{I}} R_{12} = \varphi_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}} R_{21} R_{12},$$

where the relation (3.6b) has been used; On the other hand,

$$\begin{aligned} S\big(V_{12}^{\mathrm{II\,I}} \varphi_2^{\mathrm{II}} Z_{21} \varphi_1^{\mathrm{I}} R_{12}\big) &= \cdot\Psi\big(V_{12}^{\mathrm{II\,I}} S\big(\varphi_2^{\mathrm{II}}\big) \otimes Z_{21} S\big(\varphi_1^{\mathrm{I}}\big) R_{12}\big) \\ &= V_{12}^{\mathrm{II\,I}} \bar{V}_{21}^{\mathrm{II\,I}} \varphi_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}} R_{21} R_{12} = \varphi_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}} R_{21} R_{12}. \end{aligned}$$

So we have $S(\varphi_1^{\mathrm{I}} Z_{12} \varphi_2^{\mathrm{II}}) = S(V_{12}^{\mathrm{II\,I}} \varphi_2^{\mathrm{II}} Z_{21} \varphi_1^{\mathrm{I}} R_{12})$. On the high-order products and the other braidings, the calculations are similar. Thus S is well defined.

Other axioms such as $\cdot(S \otimes id)\Delta = \cdot(id \otimes S)\Delta = \eta\varepsilon$, etc., are easily verified.

Therefore, equipped with coproduct (3.1), counit, antipode (3.2), and one of the braidings in (3.3), the differential algebra $\Omega_{A(Q,R,Z)}$ becomes a braided Hopf algebra, this braided (additive) differential Hopf algebra will be denoted by $\underline{\Omega}_{A(Q,R,Z)}$. □

4. SPECIAL CASES

The braided (additive) differential Hopf algebra $\underline{\Omega}_{A(Q,R,Z)}$ given in Section 3 contains plenty of special cases, some of which are well known. These special cases can be obtained from $\underline{\Omega}_{A(Q,R,Z)}$ by suitable choice of the R-matrix triple (Q, R, Z). (In this section Z is also supposed to be invertible).

Example 4.1. Taking $Q_{12} = Z_{12} R_{21} Z_{21}^{-1}$, then from Proposition 2.1, the R-matrix triple (Q, R, Z) is reduced to an R-matrix pair (R, Z) satisfying (2.1), and the algebra relations (2.6) are reduced to (2.4). Correspondingly, the $\underline{\Omega}_{A(Q,R,Z)}$ is reduced to $\underline{\Omega}_{A(R,Z)}$: a braided (additive) differential Hopf algebra based on the $\Omega_{A(R,Z)}$ given by (2.4). Explicitly, $\underline{\Omega}_{A(R,Z)}$ has the algebra relations (2.4), the additive coproduct (3.1), the counit, antipode (3.2), and one of the following four braidings obtained from (3.3) as

$$\begin{aligned}
\tilde{T}_1 Z_{12} T_2 &= Z_{12} R_{21} Z_{21}^{-1} T_2 Z_{21} \tilde{T}_1 R_{12}, \\
\tilde{T}_1 Z_{12} dT_2 &= Z_{12} R_{21} Z_{21}^{-1} dT_2 Z_{21} \tilde{T}_1 R_{12} + \lambda P_{12} T_2 Z_{21} d\tilde{T}_1 R_{12}, \\
d\tilde{T}_1 Z_{12} T_2 &= Z_{12} R_{12}^{-1} Z_{21}^{-1} T_2 Z_{21} d\tilde{T}_1 R_{12}, \\
d\tilde{T}_1 Z_{12} dT_2 &= -Z_{12} R_{12}^{-1} Z_{21}^{-1} dT_2 Z_{21} d\tilde{T}_1 R_{12};
\end{aligned} \tag{4.1a}$$

$$\begin{aligned}
\tilde{T}_1 Z_{12} T_2 &= Z_{12} R_{12}^{-1} Z_{21}^{-1} T_2 Z_{21} \tilde{T}_1 R_{12}^{-1}, \\
\tilde{T}_1 Z_{12} dT_2 &= Z_{12} R_{21} Z_{21}^{-1} dT_2 Z_{21} \tilde{T}_1 R_{12}^{-1}, \\
d\tilde{T}_1 Z_{12} T_2 &= Z_{12} R_{12}^{-1} Z_{21}^{-1} T_2 Z_{21} d\tilde{T}_1 R_{12}^{-1} - \lambda P_{12} dT_2 Z_{21} \tilde{T}_1 R_{12}^{-1}, \\
d\tilde{T}_1 Z_{12} dT_2 &= -Z_{12} R_{21} Z_{21}^{-1} dT_2 Z_{21} d\tilde{T}_1 R_{12}^{-1};
\end{aligned} \tag{4.1b}$$

$$\begin{aligned}
\tilde{T}_1 Z_{12} T_2 &= Z_{12} R_{21} Z_{21}^{-1} T_2 Z_{21} \tilde{T}_1 R_{12}, \\
\tilde{T}_1 Z_{12} dT_2 &= Z_{12} R_{21} Z_{21}^{-1} dT_2 Z_{21} \tilde{T}_1 R_{12} + \lambda Z_{12} R_{21} Z_{21}^{-1} T_2 Z_{21} d\tilde{T}_1 P_{12}, \\
d\tilde{T}_1 Z_{12} T_2 &= Z_{12} R_{21} Z_{21}^{-1} T_2 Z_{21} d\tilde{T}_1 R_{21}^{-1}, \\
d\tilde{T}_1 Z_{12} dT_2 &= -Z_{12} R_{21} Z_{21}^{-1} dT_2 Z_{21} d\tilde{T}_1 R_{21}^{-1};
\end{aligned} \tag{4.1c}$$

$$\begin{aligned}
\tilde{T}_1 Z_{12} T_2 &= Z_{12} R_{12}^{-1} Z_{21}^{-1} T_2 Z_{21} \tilde{T}_1 R_{21}^{-1}, \\
\tilde{T}_1 Z_{12} dT_2 &= Z_{12} R_{12}^{-1} Z_{21}^{-1} dT_2 Z_{21} \tilde{T}_1 R_{12}, \\
d\tilde{T}_1 Z_{12} T_2 &= Z_{12} R_{12}^{-1} Z_{21}^{-1} T_2 Z_{21} d\tilde{T}_1 R_{21}^{-1} - \lambda Z_{12} R_{12}^{-1} Z_{21}^{-1} dT_2 Z_{21} \tilde{T}_1 P_{12}, \\
d\tilde{T}_1 Z_{12} dT_2 &= -Z_{12} R_{12}^{-1} Z_{21}^{-1} dT_2 Z_{21} d\tilde{T}_1 R_{12}.
\end{aligned} \tag{4.1d}$$

Example 4.2. Taking $Q_{12} = R_{12} = Z_{12}$, then the R-matrix triple (Q, R, Z) is reduced to a single matrix solution R of the QYBE, and (2.6) is reduced to the algebra relations of the differential complex $\Omega_{B(R)}$ on the braided matrix algebra $B(R)$ (Azcarraga *et al.*, 1994, Kulish and Sasaki, 1993; Kulish and Sklyanin, 1992; Majid, 1991, 1993; Ogievetsky *et al.*, 1992):

$$\begin{aligned}
R_{21} T_1 R_{12} T_2 &= T_2 R_{21} T_1 R_{12}, \\
R_{12}^{-1} T_1 R_{12} dT_2 &= dT_2 R_{21} T_1 R_{12}, \\
R_{12}^{-1} dT_1 R_{12} dT_2 &= -dT_2 R_{21} dT_1 R_{12}.
\end{aligned} \tag{4.2}$$

Correspondingly, the $\underline{\Omega}_{A(Q,R,Z)}$ is reduced to $\underline{\Omega}_{B(R)}$: a braided (additive) differential Hopf algebra on $\Omega_{B(R)}$. Concretely, the $\underline{\Omega}_{B(R)}$ has the algebra relations (4.2), the additive coproduct (3.1), the counit, antipode (3.2) and one of the following four braidings obtained from (3.3) as

$$\begin{aligned}
\tilde{T}_1 R_{12} T_2 &= R_{12} T_2 R_{21} \tilde{T}_1 R_{12},\\
\tilde{T}_1 R_{12} dT_2 &= R_{12} dT_2 R_{21} \tilde{T}_1 R_{12} + \lambda P_{12} T_2 R_{21} d\tilde{T}_1 R_{12},\\
d\tilde{T}_1 R_{12} T_2 &= R_{21}^{-1} T_2 R_{21} d\tilde{T}_1 R_{12},\\
d\tilde{T}_1 R_{12} dT_2 &= -R_{21}^{-1} dT_2 R_{21} d\tilde{T}_1 R_{12};
\end{aligned} \tag{4.3a}$$

$$\begin{aligned}
\tilde{T}_1 R_{12} T_2 &= R_{21}^{-1} T_2 R_{21} \tilde{T}_1 R_{12}^{-1},\\
\tilde{T}_1 R_{12} dT_2 &= R_{12} dT_2 R_{21} \tilde{T}_1 R_{12}^{-1},\\
d\tilde{T}_1 R_{12} T_2 &= R_{21}^{-1} T_2 R_{21} d\tilde{T}_1 R_{12}^{-1} - \lambda P_{12} dT_2 R_{21} \tilde{T}_1 R_{12}^{-1},\\
d\tilde{T}_1 R_{12} dT_2 &= -R_{12} dT_2 R_{21} d\tilde{T}_1 R_{12}^{-1};
\end{aligned} \tag{4.3b}$$

$$\begin{aligned}
\tilde{T}_1 R_{12} T_2 &= R_{12} T_2 R_{21} \tilde{T}_1 R_{12},\\
\tilde{T}_1 R_{12} dT_2 &= R_{12} dT_2 R_{21} \tilde{T}_1 R_{12} + \lambda R_{12} T_2 R_{21} d\tilde{T}_1 P_{12},\\
d\tilde{T}_1 R_{12} T_2 &= R_{12} T_2 R_{21} d\tilde{T}_1 R_{21}^{-1},\\
d\tilde{T}_1 R_{12} dT_2 &= -R_{12} dT_2 R_{21} d\tilde{T}_1 R_{21}^{-1};
\end{aligned} \tag{4.3c}$$

$$\begin{aligned}
\tilde{T}_1 R_{12} T_2 &= R_{21}^{-1} T_2 R_{21} \tilde{T}_1 R_{21}^{-1},\\
\tilde{T}_1 R_{12} dT_2 &= R_{21}^{-1} dT_2 R_{21} \tilde{T}_1 R_{12},\\
d\tilde{T}_1 R_{12} T_2 &= R_{21}^{-1} T_2 R_{21} d\tilde{T}_1 R_{21}^{-1} - \lambda R_{21}^{-1} dT_2 R_{21} \tilde{T}_1 P_{12},\\
d\tilde{T}_1 R_{12} dT_2 &= -R_{21}^{-1} dT_2 R_{21} d\tilde{T}_1 R_{12}.
\end{aligned} \tag{4.3d}$$

The above $\underline{\Omega}_{B(R)}$ is, in fact, a "right-hand" version of the related result given by Isaev and Vladimirov (1995) and Vladimirov (1994).

Example 4.3. Taking $Q_{12} = R_{21}$, $Z = 1$, then (2.5) is reduced essentially to the QYBE about R, and (2.6) is reduced to the algebra relations of the differential complex $\Omega_{A(R)}$ on the quantum matrix algebra $A(R)$ (Faddeev *et al.*, 1989; Sudbery, 1992, 1993):

$$\begin{aligned}
R_{12} T_1 T_2 &= T_2 T_1 R_{12},\\
R_{21}^{-1} T_1 dT_2 &= dT_2 T_1 R_{12},\\
R_{21}^{-1} dT_1 dT_2 &= -dT_2 dT_1 R_{12}.
\end{aligned} \tag{4.4}$$

Correspondingly, the $\underline{\Omega}_{A(Q,R,Z)}$ is reduced to $\underline{\Omega}_{A(R)}$: a braided (additive) differential Hopf algebra on $\Omega_{A(R)}$. Explicitly, the $\underline{\Omega}_{A(R)}$ has the algebra relations (4.4), the additive coproduct (3.1), the counit, antipode (3.2), and one of the following four braidings obtained from (3.3) as

$$\begin{aligned}
\tilde{T}_1 T_2 &= R_{21} T_2 \tilde{T}_1 R_{12}, \\
\tilde{T}_1 dT_2 &= R_{21} dT_2 \tilde{T}_1 R_{12} + \lambda P_{12} T_2 d\tilde{T}_1 R_{12}, \\
d\tilde{T}_1 T_2 &= R_{12}^{-1} T_2 d\tilde{T}_1 R_{12}, \\
d\tilde{T}_1 dT_2 &= -R_{12}^{-1} dT_2 d\tilde{T}_1 R_{12};
\end{aligned} \tag{4.5a}$$

$$\begin{aligned}
\tilde{T}_1 T_2 &= R_{12}^{-1} T_2 \tilde{T}_1 R_{12}^{-1}, \\
\tilde{T}_1 dT_2 &= R_{21} dT_2 \tilde{T}_1 R_{12}^{-1}, \\
d\tilde{T}_1 T_2 &= R_{12}^{-1} T_2 d\tilde{T}_1 R_{12}^{-1} - \lambda P_{12} dT_2 \tilde{T}_1 R_{12}^{-1}, \\
d\tilde{T}_1 dT_2 &= -R_{21} dT_2 d\tilde{T}_1 R_{12}^{-1};
\end{aligned} \tag{4.5b}$$

$$\begin{aligned}
\tilde{T}_1 T_2 &= R_{21} T_2 \tilde{T}_1 R_{12}, \\
\tilde{T}_1 dT_2 &= R_{21} dT_2 \tilde{T}_1 R_{12} + \lambda R_{21} T_2 d\tilde{T}_1 P_{12}, \\
d\tilde{T}_1 T_2 &= R_{21} T_2 d\tilde{T}_1 R_{21}^{-1}, \\
d\tilde{T}_1 dT_2 &= -R_{21} dT_2 d\tilde{T}_1 R_{21}^{-1};
\end{aligned} \tag{4.5c}$$

$$\begin{aligned}
\tilde{T}_1 T_2 &= R_{12}^{-1} T_2 \tilde{T}_1 R_{21}^{-1}, \\
\tilde{T}_1 dT_2 &= R_{12}^{-1} dT_2 \tilde{T}_1 R_{12}, \\
d\tilde{T}_1 T_2 &= R_{12}^{-1} T_2 d\tilde{T}_1 R_{21}^{-1} - \lambda R_{12}^{-1} dT_2 \tilde{T}_1 P_{12}, \\
d\tilde{T}_1 dT_2 &= -R_{12}^{-1} dT_2 d\tilde{T}_1 R_{12}.
\end{aligned} \tag{4.5d}$$

The $\underline{\Omega}_{A(R)}$ here is in reality a "right-hand" version of the corresponding result given by Vladimirov (1994).

There are also other special cases of $\underline{\Omega}_{A(Q,R,Z)}$ such as braided (additive) differential Hopf algebras on the quantum supermatrix algebras (Liao and Song, 1991; Manin, 1989), on the quantum anyonic matrix algebras (Majid and Rodriguez-Plaza, 1994), and on the μ-braided GL_q (Couture and Leivo, 1994), etc.; these can all be obtained from $\underline{\Omega}_{A(Q,R,Z)}$ by choosing suitable R-matrix triple (Q, R, Z), but here we have not discussed them in any detail.

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